1. Consider the region bounded by the graphs of \( f(x) = \frac{8x}{x+1} \), \( x=0 \), \( x=4 \) and \( y=0 \) as shown in the figure below. Draw and use rectangles to find a Left Riemann Sum to approximate the area of the region.

\[
\int_{0}^{4} f(x) \, dx \approx f(0) + f(1) + f(2) + f(3)
\]

\[
= 1(0) + 1(4) + 1\left(\frac{14}{3}\right) + 1(6)
\]

\[
= 10 + \frac{14}{3} + 6
\]

\[
= \frac{46}{3}
\]

\[
\approx 15.333
\]

2. Consider the region bounded by the graphs of \( f(x) = \frac{8x}{x+1} \), \( x=0 \), \( x=4 \) and \( y=0 \) as shown in the figure below. Draw and use rectangles to find a Right Riemann Sum to approximate the area of the region.

\[
\int_{0}^{4} f(x) \, dx \approx f(1) + f(2) + f(3) + f(4)
\]

\[
= 4 + \frac{14}{3} + 6 + \frac{32}{5}
\]

\[
= \frac{46}{3} + \frac{32}{5}
\]

\[
= \frac{230}{15} + \frac{96}{15} = \frac{326}{15}
\]

\[
\approx 21.733
\]

3. Consider the region bounded by the graphs of \( f(x) = \frac{8x}{x+1} \), \( x=0 \), \( x=4 \) and \( y=0 \) as shown in the figure below. Draw and use four midpoint rectangles to find the approximate area of the region.

\[
\int_{0}^{4} f(x) \, dx \approx f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{7}{2}\right)
\]

\[
= 2.667 + 4.8 + 5.714 + 6.222
\]

\[
\approx 19.403
\]
4. Draw and use four trapezoids and the trapezoidal rule to find the approximate area under the curve bounded by the graphs of \( f(x) = \frac{8x}{x+1} \), \( x=0 \), \( x=4 \) and \( y=0 \) as shown in the figure below. Is there a more accurate method of finding the area? If so, what is it?

\[
\int_{0}^{4} \frac{8x}{x+1} \, dx \approx \frac{4-0}{2(4)} \left[ f(0) + 2f(1) + 2f(2) + 2f(3) + f(4) \right]
\]

\[
= \frac{1}{2} \left[ 0 + 8 + \frac{16}{3} + 2(6) + \frac{32}{5} \right]
\]

\[
= \frac{1}{2} \left[ 0 + 8 + \frac{16}{3} + 12 + \frac{32}{5} \right]
\]

\[
\approx \frac{1}{2} \left[ 20 + \frac{160}{15} \right]
\]

\[
\approx 18.533
\]

5. Sketch the graph of \( f(x) = x^2 - 6x + 13 \).

6. On the graph above, shade the region between the function, \( y = 0 \), \( x = 2 \) and \( x = 5 \).

7. Use antiderivatives to find the EXACT area under the curve in problem number 5.

\[
\int_{2}^{5} (x^2 - 6x + 13) \, dx = \left[ \frac{x^3}{3} - 3x^2 + 13x \right]_{2}^{5}
\]

\[
= 15
\]

8. Find the average value of the function given in problem #5 on the interval from 2 to 5.

\[
\frac{1}{5-2} \int_{2}^{5} (x^2 - 6x + 13) \, dx = \frac{1}{3} (15) = 5
\]
Chapter 4 Review

Find the value of each definite integral for problems 9 – 12 below.

9. \[ \int_{0}^{1} \frac{x - \sqrt{x}}{3x} \, dx = \int_{0}^{1} \frac{1}{3} \, dx - \frac{1}{3} \int_{0}^{1} x^{-\frac{1}{2}} \, dx \]
   \[ = \left[ \frac{1}{3} x \right]_{0}^{1} - \frac{2}{3} \left[ x^{\frac{1}{2}} \right]_{0}^{1} \]
   \[ = \frac{1}{3} - 0 - \frac{2}{3} \left( 1 - 0 \right) = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3} \]

10. \[ \int_{0}^{\pi} (1 + \sin x) \, dx \]
    \[ = \left[ x - \cos x \right]_{0}^{\pi} \]
    \[ = (\pi - \cos \pi) - (0 - \cos 0) \]
    \[ = \pi + 1 \]

11. \[ \int_{1}^{4} x(x^2 + 1)^3 \, dx \]
    Let \( u = x^2 + 1 \), then \( du = 2x \, dx \)
    \[ \frac{1}{2} \int_{1}^{4} u^3 \, du = \frac{1}{2} \left[ \frac{1}{4} (u^4) \right]_{1}^{4} \]
    \[ = \frac{1}{8} (25) - \frac{1}{8} (2) = \frac{23}{8} \]

12. \[ \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \csc x \cot x \, dx \]
    \[ = \left[ -\csc x \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \]
    \[ = \csc \frac{\pi}{6} - \csc \frac{\pi}{3} = \frac{2}{3} + 2 \]
    \[ = \frac{8}{3} \]

13. Find the antiderivative of \( x^2(x^3 - 4)^7 \).
    \[ \int x^2(x^3 - 4)^7 \, dx \]
    Let \( u = x^3 - 4 \), then \( du = 3x^2 \, dx \)
    \[ \int x^2 \cdot u^7 \, du \]
    \[ = \frac{1}{3} \int u^7 \, du \]
    \[ = \frac{u^8}{24} + C = \left( \frac{x^3 - 4}{24} \right)^8 + C \]

14. Evaluate \( \int x\sqrt{x + 3} \, dx \)
    \[ \int \left( u - 3 \right) \, du \]
    \[ = \frac{u^2}{2} + C = \frac{(x + 3)^2}{2} + C \]
    \[ \int_{2}^{4} \left( u^3 - 3u \right) \, du \]
    \[ = \left[ \frac{2}{5} u^{\frac{5}{2}} - 3u^2 \right]_{2}^{4} = 18.01 \]
15. The table below shows the velocity of a remote-controlled toy car as it traveled down a hallway for 10 seconds.
   a) Use midpoint Riemann sums with 5 subintervals of equal length to approximate $\int v(t)\,dt$. Indicate units of measure in your answer.

   \[
   \int v(t)\,dt \approx 2(f(1) + f(3) + f(5) + f(7) + f(9)) \\
   = 2(6 + 10 + 16 + 12 + 4) = 2(60) = 120 \text{ inches}
   \]

<table>
<thead>
<tr>
<th>$t$ (seconds)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(t)$ (inches per second)</td>
<td>0</td>
<td>6</td>
<td>10</td>
<td>16</td>
<td>14</td>
<td>12</td>
<td>18</td>
<td>22</td>
<td>12</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

   b) What is the meaning of your answer to part (a)?

   The distance the car travelled is about 120 inches.

16. The graph of $f(t)$ is shown below.

   ![Graph of f(t)]

   (a) If $F(x) = \int_0^x f(t)\,dt$, fill in the values for $F(x)$ in the table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F(x)$</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>$-\frac{1}{2}$</td>
</tr>
</tbody>
</table>

   (b) Calculate and sketch the average value of $f(t)$ on the graph above.

   \[
   \frac{1}{6} \int_0^6 f(x)\,dx = \frac{1}{6} \cdot \frac{1}{2} = -\frac{1}{12}
   \]
Chapter 4 Review

17. Use the Second Fundamental Theorem of Calculus to find $F'(2)$ if

$$F(x) = \int^x_1 \sqrt{t^2 + 3} \, dt$$

$$F'(x) = \sqrt{(x^2)^2 + 3} \cdot (2x)$$

$$F'(2) = 4 \sqrt{7}$$

18. Solve the differential equation $\frac{dy}{dx} = 2xy^2$ when $y = -1$ and $x = 1$.

$$\int \frac{dy}{y^2} = \int 2x \, dx$$

$$\int y^{-2} \, dy = 2\int x \, dx$$

$$-\frac{1}{y} = \frac{x^2}{2} + C$$

$$-\frac{1}{1} = \frac{1}{4} + C$$

$$C = 0$$

$$-\frac{1}{y} = \frac{x^2}{2}$$

$$\frac{1}{y} = -\frac{x^2}{2}$$

$$y = -\frac{1}{x^2}$$

19. Consider the differential equation $\frac{dy}{dx} = \frac{-x^2y^2}{2}$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = 2$.

a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.

b) Write an equation for the line tangent to the graph of $f$ at $x = -1$. $\frac{dy}{dx} = \frac{-(-1)(2)^2}{2} = 2$

$$y - 2 = 2(x + 1)$$

\[ y = -\frac{1}{4}x^2 + C \]

$$C = \frac{1}{4}$$

\[ y = \frac{1}{4}x^2 + \frac{1}{4} \]

$$y = \frac{y}{x^2 + 1}$$

\[ y = -\frac{1}{4}x^2 - \frac{1}{4} \]

\[ y = \frac{1}{4}x + C \]

\[ y = \frac{1}{4}x^2 + \frac{1}{4} \]